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NONLINEAR EFFECTS IN THE PROBLEM OF THE BEAM ON A FOUNDATION WITH A MOVING LOAD*

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Abstract—The "steady-state" solution of the linearized equations for a beam on an elastic foundation (with no damping) with a load moving at a certain velocity, referred to as the "critical" velocity, does not exist. In this paper, suitable perturbation solutions are obtained for equations for the steady-state motion which include the geometric and material nonlinearities. Secular terms are avoided by using the usual Poincaré expansion for subcritical load speeds and a Lindstedt expansion for supercritical speeds which may be extended to the critical speed when the material nonlinearity dominates. However neither expansion is valid for the critical velocity when the geometric nonlinearity dominates, as for a very slender beam. For this situation, a successful expansion is found which gives a solution that is mainly periodic (with the distance from the load) but with a slow monotonic decrease in the envelope. The results are the first nonlinear corrections to the linear solutions for noncritical load speeds and the solution for the critical speed which gives a strain which varies with the square root of the load. To indicate the plasticity effect, a simple solution for supercritical load speeds for elastic-plastic beam and foundation materials is also obtained. The solution consists mainly of the elastic waves but with a region of plastic flow of the foundation behind the load and two points of yielding of the beam in bending ahead of the load.

NOTATION

X	distance in fixed coordinate system along undeformed beam neutral axis
V	load velocity
t	time
x	$X - vt$, distance in moving coordinate system along undeformed beam neutral axis
y	transverse displacement of beam
u	axial displacement
ψ	angle of rotation
M	moment
T	axial force
Q	transverse shear force
E, E_1	elastic constants
I_2, I_4	moment of inertia and fourth moment of beam cross-section
A	area of beam cross-section
r	$(I_2/A)^{1/2}$
k_1, k_2, k_3	foundation stiffness factors
ρ	density of beam material
N	transverse force
λ	$(EA/4k_1r^2)^{1/4}$
ξ	$x/(2^{1/2}\lambda r)$
η	$y/(2^{1/2}\lambda r)$
γ	$u/(2^{1/2}\lambda r)$
v	$\lambda V(\rho/E)^{1/2}$
α	$2\lambda^2 r^2 k_2/k_1$
β	$k_3/2k_1$
Δ	$2\lambda^2 N/EA$
μ	$E_1 I_4/2\lambda^2 r^2 E I_2$

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INTRODUCTION

THE effect of moving loads on structures has been the subject of numerous investigations. When the structure has a simple geometry and the load moves with constant velocity, a relatively simple "steady-state" solution can be obtained, as in the investigation by Kenney [1]. However, for the various beams on an elastic foundation, cylindrical shells, simply supported plate strips, etc. for which steady-state solutions have been obtained, there are certain critical velocities (usually small in comparison with the sonic velocities of the materials). For subcritical velocities the response is localized; for supercritical velocities monochromatic wave trains are generated ahead of and behind any discontinuities in the moving load distribution. At the critical velocity a steady-state solution of the linear equations does not exist unless damping is included as in [1].

Another approach is taken in [2] in which the transient solution (to the linear undamped equations) is obtained for the finite beam. For a long beam on an elastic foundation, the solution approaches the steady-state for noncritical load velocities. At the critical velocity, the response increases with the square root of the distance of the load from the end. Thus the linear equations indicate that large amplitude response is possible for a long beam on an elastic foundation with minimal damping.

For this reason, equations which retain geometric and material nonlinearities are considered in the present investigation. A steady-state solution is obtained for the critical load velocity which indicates a response proportional to the square root of the load and provides the limitation on the transient solution of the linear equations in [2].

The existence of such a nonlinear steady-state solution was indicated by Stoker in a discussion of a somewhat analogous fluid flow problem ([3] p. 217). In addition the first nonlinear effects for the noncritical load speeds are obtained in this investigation.

An interesting feature of the noncritical speeds is in the type of perturbation expansions that are required. For subcritical speed, for which the linear solution is exponential, the usual (Poincaré) expansion is quite successful. For supercritical speed, for which the linear solution is periodic, a Lindstedt expansion, discussed by Cesari [4], avoids secular terms. However, when the geometric nonlinearity dominates, neither expansion is successful for the critical speed, for which there is a secular solution to the homogeneous linearized equations. An appropriate expansion is used which gives a solution that is mainly periodic but with a slowly decreasing envelope.

For large load magnitudes, plasticity effects become important, particularly for the supercritical load speed for which the elastic solution indicates many cycles of loading for each beam element. Therefore a solution is obtained for the equations for small deformations but elastic–perfectly plastic beam and foundation. The solution satisfies all continuity conditions and seems to be reasonable. However, it may not be unique, although attempts to construct other forms of solution were not successful. In its favor, the present elastic–plastic solution approaches the elastic solution as the load decreases.

BEAM EQUATIONS

The equations for the plane motion of an extensible elastica were obtained by Tadjbakhsh [5]. To his equations terms for elastic restraint of a foundation are added. The sign convention is shown in Fig. 1. A point of the beam at $(X, 0)$ before deformation is at the position

$(X+u, y)$ after deformation. The equilibrium equations are

$$-\frac{\partial}{\partial X}(T\cos\psi+Q\sin\psi)+k_3u+\rho A\frac{\partial^2 u}{\partial t^2}=0 \tag{1a}$$

$$\frac{\partial}{\partial X}(Q\cos\psi-T\sin\psi)+k_1y-k_2y^3+\rho A\frac{\partial^2 y}{\partial t^2}=0 \tag{1b}$$

$$\frac{\partial s}{\partial X}\left(\frac{\partial M}{\partial s}-Q\right)=J\frac{\partial^2\psi}{\partial t^2} \tag{1c}$$

in which

$$\psi=\tan^{-1}\left[\left(1+\frac{\partial u}{\partial X}\right)^{-1}\frac{\partial u}{\partial X}\right]$$

$$\frac{\partial s}{\partial X}=\left[\left(1+\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial y}{\partial X}\right)^2\right]^{\frac{1}{2}}.$$

The constants k_1, k_2, and k_3 give the stiffness properties of the foundation; k_1 and k_3 are assumed to be nonzero and positive.

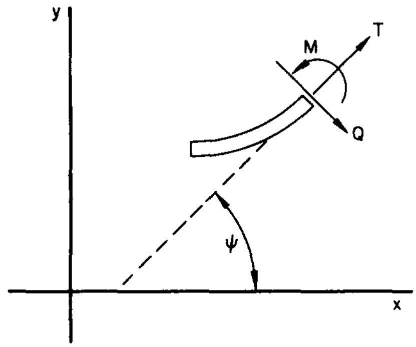

FIG. 1. Beam element.

The mass moment of inertia per unit of undeformed length is J. However, rotary inertia is significant, certainly for the linearized moving load problem, only for load speeds of the magnitude of the sonic speeds of the beam material, in which case shear deformation should also be taken into consideration. The same should be true even when the first nonlinear effects become significant. Hence, for this investigation, we take $J=0$ and consider only subsonic speeds ($v \ll \lambda$).

The deformation measures are the strain

$$e=\frac{\partial s}{\partial X}-1 \tag{1d}$$

and the dimensionless curvature

$$\tau=r\partial\psi/\partial s \tag{1e}$$

where r is chosen as the radius of gyration of the beam cross-section. The constitutive relations from [5] for an elastica are

$$M = \frac{r}{1+e}\frac{\partial W}{\partial \tau} \tag{1f}$$

$$T = \frac{\partial W}{\partial e} - \frac{\tau}{1+e}\frac{\partial W}{\partial \tau} \tag{1g}$$

where W is, from the variational formulation, an indeterminate function of e and τ. The selection of a suitable form for W can be guided by consideration of a direct simplistic approach. A typical material has the stress–strain relation $\sigma = Ee - E_1 e^3 + O(e^5)$ in which E_1/E is quite large, say $O(10^3)$, so that the material nonlinearity becomes important while the strain e is still small. For a beam in combined bending and elongation the fiber stress would be, if the geometric nonlinearities and the strains $O(e^5)$ are ignored,

$$\sigma = E(e + \tau z/r) - E_1(e + \tau z/r)^3$$

where z is the distance from the centroid of the cross-section. The stress resultants are then

$$M = \int \sigma z \, \mathrm{d}z = EAr\left[\tau - \frac{E_1 I_4}{EAr^4}\tau^3 - 3\frac{E_1}{E}e^2\tau\right]$$

$$T = \int \sigma \, \mathrm{d}z = EA\left[e - (e^3 + 3e\tau^2)E_1/E\right].$$

So an appropriate choice for W seems to be

$$W = EA\left\{\frac{e^2}{2} + (1+e)^2\left[\frac{\tau^2}{2} - \frac{E_1 I_4}{4EAr^4}\tau^4 - \frac{3E_1 e^2\tau^2}{2E}\right] - \frac{E_1 e^4}{4E}\right\}$$

which used in (1f) and (1g) gives

$$M = EAr(1+e)\left[\tau - \frac{E_1 I_4}{EAr^4}\tau^3 - \frac{3E_1 e^2 \tau}{E}\right] \tag{2a}$$

$$T = EA\left[e - \frac{E_1}{E}e^3 - \frac{3E_1}{E}e(1+e)^2\tau^2 + \frac{E_1 I_4}{2EAr^4}(1+e)\tau^4\right] \tag{2b}$$

which coincide with the results of the direct approach when $e^2, \tau^2 \ll 1$. Hence (2a) and (2b) will be regarded as appropriate constitutive relations for an elastica with constants selected to relate the elastica with the material and shape properties of a three-dimensional beam.

In this investigation the solution is sought for a concentrated load N, normal to the undeformed beam, moving at a constant velocity V. Hence the transverse and tangential forces are discontinuous at the load

$$Q|_{X=Vt+} - Q|_{X=Vt-} = N\cos\psi \tag{3a}$$

$$T|_{X=Vt+} - T|_{X=Vt-} = -N\sin\psi. \tag{3b}$$

We will seek the "steady-state" solution which is a solution depending only on the distance from the moving load $x = X - Vt$. Thus

$$y = y(x), \qquad u = u(x).$$

Since the present interest is in the first nonlinear effects, only the first nonlinear terms will be retained in the equations. For normal loading of a beam it is easy to see that $e = O(\tau^2)$ which, in terms of the dimensionless variables defined in Notation, gives $\gamma = O(\eta^2)$. Hence the appropriate expansions are

$$\begin{aligned}
e &= \gamma' + \tfrac{1}{2}(\eta')^2 + O(\eta^4) \\
\psi &= \eta' - [\eta'\gamma' + \tfrac{1}{3}(\eta')^3] + O(\eta^5) \\
2^{\frac{1}{2}}\lambda\tau &= \eta'' - [\eta''(2\gamma' + \tfrac{3}{2}(\eta')^2) + \eta'\gamma''] + O(\eta^5) \\
\frac{2^{\frac{1}{2}}\lambda M}{EAr} &= \eta'' - [\eta''\gamma' + \eta'\gamma'' + (\eta')^2\eta'' + \mu(\eta'')^3] + O(\eta^5) \\
\frac{T}{EA} &= \gamma' + \tfrac{1}{2}(\eta')^2 + O(\eta^4) \\
\frac{2\lambda^2 Q}{EA} &= \eta'''[\eta'''2(\gamma' + (\eta')^2) + 2\eta''(\gamma'' + \eta'\eta'') + \eta'\gamma''' + \mu((\eta'')^3)'] + O(\eta^5)
\end{aligned} \tag{4}$$

in which the primes denote differentiation with respect to ξ. The equations (1a) and (1b) become

$$(\lambda^2 - v^2)\gamma'' - \beta\gamma = -\tfrac{1}{2}[\lambda^2(\eta')^2 + \eta'\eta''']' + O(\eta^4) \tag{5a}$$

$$\begin{aligned}
&\eta'''' - [\eta''''(2\gamma' + \tfrac{5}{2}(\eta')^2) + \eta'''(3\gamma'' + 8\eta'\eta'') + \eta''(2\gamma''' + (\eta'')^2) + \gamma''''\eta' + \mu((\eta'')^3)'']\\
&+ 2v^2\eta'' + \eta - \alpha\eta^3 - 2\lambda^2[\eta'(\gamma' + \tfrac{1}{2}(\eta')^2)]' + O(\eta^5) = 0.
\end{aligned} \tag{5b}$$

The solution of (5) is sought for which η and γ are bounded for $|\xi| \to \infty$, and for which $\gamma, \eta, \eta', \eta''$ are continuous at $\xi = 0$. For the typical practical problem, the parameters λ, α, and μ are large in comparison with 1 while β is around 1. The load parameter Δ could, of course, be of arbitrary magnitude. It would be expected that the linear theory would be valid for sufficiently small Δ which should give a small amplitude of η. Thus all the expansions are in powers of Δ. After the formulas for the constants of the expansions are obtained, the fact that $\lambda, \alpha, \mu \gg 1$ is used for simplification.

LINEAR SOLUTION

When the load is sufficiently small $\Delta \lll 1$, then the solution η should also be small. The omission of the nonlinear terms of (5) gives

$$\begin{aligned}
\gamma &= O(\eta^2) \\
\eta'''' + 2v^2\eta'' + \eta &= 0.
\end{aligned} \tag{6}$$

For $v < 1$ the solution of (6) which is bounded for $|\xi| \to \infty$ is

$$\eta = \frac{\Delta}{2(1-v^4)^{\frac{1}{2}}} e^{-|\xi|\sin\varphi} \cos(|\xi|\cos\varphi - \varphi) \tag{7a}$$

where

$$\varphi = \cos^{-1}[(1+v^2)/2]^{\frac{1}{2}} \qquad (0 \leq \varphi \leq \pi/4)$$

The maximum curvature change is given by

$$\tau_0 = \frac{\eta''(0)}{2^{\frac{1}{2}}\lambda} = \frac{\Delta}{4\lambda(1-v^2)^{\frac{1}{2}}}. \tag{7b}$$

For $v > 1$, the complementary solutions of (6) are periodic. The unique steady-state solution is obtained by either adding a small damping term, as in [1], or by considering the transient solution for a semi-infinite beam, as in [2]. Both approaches indicate that the shorter wavelength "wave train" is ahead of the load

$$\eta = \frac{\Delta}{2(v^4-1)^{\frac{1}{2}}} \times \begin{cases} -\omega_b^{-1}\sin\omega_b\xi & \text{for} \quad \xi > 0 \\ -\omega_a^{-1}\sin\omega_a\xi & \text{for} \quad \xi < 0 \end{cases} \tag{8a}$$

where

$$\omega_a = v[1-(1-v^{-4})^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\omega_a^{-1} = \omega_b = v[1+(1-v^{-4})^{\frac{1}{2}}]^{\frac{1}{2}} \approx 2^{\frac{1}{2}}v \qquad \text{for} \quad v \gtrsim 2.$$

The maximum bending strain is given by

$$\tau_0 = \frac{\eta''(\pi/2)}{2^{\frac{1}{2}}\lambda} = \frac{2^{\frac{1}{2}}\Delta\omega_b}{4\lambda(v^4-1)^{\frac{1}{2}}}. \tag{8b}$$

Thus $v = 1$ gives the "critical" load velocity at which no steady-state solution, bounded at $|\xi| \to \infty$, of the linear equations exists. A steady-state solution always exists if a nonzero viscous damping term is retained [1]; the amplitude becomes large as the damping becomes small. For the undamped case, the transient solution for the semi-infinite beam increases in amplitude with the square root of the distance of the load from the beam end [2]. So either approach to the linear problem leads to the possibility of arbitrarily large deformations even for small values of Δ.

PERTURBATION SOLUTION FOR SUBCRITICAL SPEED

For subcritical load speed $v < 1$ the solution of the nonlinear system (5) can be obtained by the usual (Poincaré) perturbation expansion. The solution is assumed to be expressible in the form

$$\eta = \Delta\eta_0 + \Delta^3\eta_1 + \Delta^5\eta_2 + \cdots \tag{9a}$$

$$\gamma = \Delta^2\gamma_1 + \Delta^4\gamma_2 + \cdots \tag{9b}$$

where $\Delta\eta_0$ is the linear solution (7) and the η_i an γ_i are to be determined. The series (9) are substituted into (5) and the coefficients of each power of Δ are equated to zero. Thus (5a) gives, for $\xi > 0$,

$$(\lambda^2 - v^2)\gamma_1'' - \beta\gamma_1 = -\tfrac{1}{2}[\lambda^2(\eta_0')^2 + \eta'\eta_0''']'$$
$$= -\tfrac{1}{2}e^{-2\xi\sin\varphi}[-(\lambda^2 - v^2)\sin\varphi$$
$$+ \lambda^2 \sin(2\xi\cos\varphi + \varphi) - \sin(2\xi\cos\varphi + 3\varphi)]$$

which has the solution

$$\gamma_1(\xi) = \frac{(\lambda^2 - v^2)\sin\varphi}{2(4\sin^2\varphi(\lambda^2 - v^2) - \beta)}[e^{-2\xi\sin\varphi} - e^{-\varkappa\xi}]$$
$$+ \mathscr{I}\frac{\lambda^2 e^{i\varphi} - e^{3i\varphi}}{2(4e^{2i\varphi}(\lambda^2 - v^2) + \beta)}[e^{2ie^{i\varphi}\xi} - e^{-\varkappa\xi}]$$

where

$$\varkappa = [\beta/(\lambda^2 - v^2)]^{\frac{1}{2}}$$

Since $\eta_0(-\xi) = \eta_0(\xi)$, $\gamma_1(\xi)$ will be an odd function. Thus the complementary solution has been chosen so that $\gamma_1(0) = 0$. For $\lambda \gg v, \beta$, the expression for the axial strain (2b) is

$$\frac{T}{EA} = \Delta^2[\gamma_1' + \tfrac{1}{2}(\eta_0')^2] + O(\Delta^4)$$
$$= \frac{\Delta^2\beta^{\frac{1}{2}}\cos^2\varphi}{8\lambda\sin\varphi}e^{-\beta^{\frac{1}{2}}\xi/\lambda}[1 + O(\lambda^{-1})] + O(\Delta^4) \tag{10}$$

which satisfies the condition (3b). The equation for η_1 follows from (5b)

$$\eta_1'''' + 2v^2\eta_1'' + \eta_1 = \eta_0''''(2\gamma_1' + \tfrac{5}{2}(\eta_0')^2) + \cdots + 3\mu(\eta_0'')^3)$$
$$+ \alpha\eta_0^3 + 2\lambda^2[\eta_0'(\gamma_1' + \tfrac{1}{2}(\eta_0')^2)]'$$

Since $\eta_0(\xi)$ is even, $\eta_1(\xi)$ is also even, and must satisfy the condition $\eta_1'(0) = 0$. Such a solution is readily obtained with the result that the terms multiplied by the dominant large parameter terms in the third derivative are

$$\eta_1'''(0) = -\frac{\lambda\beta^{\frac{1}{2}}\cos\varphi\cos 3\varphi}{32\sin^2\varphi} + \frac{9\mu}{4}\mathscr{R}\left\{-\frac{ie^{-2i\varphi}(2e^{2i\varphi} - 1)^2}{4(e^{2i\varphi} - 1)} - \frac{i9e^{8i\varphi}}{9e^{4i\varphi} - 1}\right\}$$
$$+ \frac{\alpha}{4}\mathscr{R}\left\{-\frac{3ie^{-2i\varphi}(2e^{2i\varphi} - 1)}{4(e^{2i\varphi} - 1)} - \frac{3i}{9e^{4i\varphi} - 1}\right\} + C\eta_0'''(0)$$

where C, the coefficient of the complementary solution, is selected to make the coefficient of Δ^3, obtained when the expression for Q (4) is substituted into the discontinuity condition (3a), equal to zero

$$C\eta_0'''(0) = [-\eta_1''' + C\eta_0''' + \eta_0'''2\gamma_1' + \eta_0''\gamma_1' + 3\mu(\eta_0'')^2\eta_0''']_{\xi=0}.$$

The maximum curvature change is

$$\tau_{max} = \tau(0) = \frac{1}{2^{\frac{1}{2}}\lambda}[\Delta\eta_0''+\Delta^3(\eta_1''-\eta_0''2\gamma_1')]_{\xi=0}$$

$$= \tau_0\left[1+\tau_0^2\frac{2\lambda^2}{(\eta_0'')^2}\left(\frac{\eta_1''}{\eta_0''}-2\gamma_1'\right)\right]_{\xi=0}$$

where τ_0 is the linear result (7b).

As the speed approaches critical, $v \to 1$, and $e^{2i\varphi} \to i(1-v^4)^{\frac{1}{2}}+1$, so that

$$\eta_1''''(0) \to -\frac{\lambda\beta^{\frac{1}{2}}}{16(1-v^2)}-\frac{9\mu+3\alpha}{16(1-v^4)^{\frac{1}{2}}}+C\eta_0'''(0)$$

For the second derivative the result is

$$\eta_1''(0) \to -\frac{\lambda(2\beta)^{\frac{1}{2}}}{16(1-v^2)^{\frac{1}{2}}}+\mu O(1)+\alpha O(1)+C\eta_0''(0)$$

where the terms $O(1)$ are bounded as $v \to 1$, so $\eta_1'''(0) \gg \eta_1''(0)$ as $v \to 1$. Thus for $v \to 1$

$$\tau_{max} \to \tau_0[1+\tau_0^2(2\lambda^3\beta^{\frac{1}{2}}+6\alpha\lambda^2[(1-v^2)/2]^{\frac{1}{2}}+6\mu\lambda^2)] \tag{11}$$

Even for v nearer to zero than 1, the result (11) retains a qualitative validity. Thus for $0 \leq v \lesssim 0{\cdot}9$ the beam and foundation material nonlinearities, given by μ and α, have about the expected effect, i.e. for a given τ_0 the increase in strain from the linear result is about equal to the deviation in the actual from the linear stress–strain curve. The geometric nonlinearities contribute the $\beta^{\frac{1}{2}}$ term to (11). For $v \geq 1$ the expansion is invalid.

PERTURBATION SOLUTION FOR SUPERCRITICAL SPEED

For supercritical load speeds $v > 1$, the Poincaré expansion (9), in which the leading term is the linear solution, yields secular terms in the successive terms. Secular terms can be avoided by using a Lindstedt expansion, discussed by Cesari [4], which has the feature that each term of the expansion has the same period. The appropriate expansion is

$$\begin{aligned}\eta &= \Delta A_1 \sin(\omega-\delta)\xi+\Delta^3A_3\sin 3(\omega-\delta)\xi+\Delta^5A_5\sin 5(\omega-\delta)\xi+\cdots\\ \gamma &= \Delta^2A_2\sin 2(\omega-\delta)\xi+\Delta^4A_4\sin 4(\omega-\delta)\xi+\cdots+Be^{-\kappa|\xi|}+\cdots\end{aligned} \tag{12}$$

where ω is the wavenumber of the linear solution (8a), i.e.

$$\omega = \begin{cases}\omega_b & \text{for} \quad \xi > 0\\ \omega_a & \text{for} \quad \xi < 0\end{cases}$$

and δ and the A_i are functions of the amplitude parameter Δ which have the expansions

$$\delta = \Delta^2\delta_1+\Delta^4\delta_2+\cdots$$

$$A_i = A_{i0}+\Delta^2A_{i1}+\Delta^4A_{i2}+\cdots$$

in which δ_j and A_{ij} are independent of Δ. The leading term A_{10} is the coefficient of the linear solution (8a)

$$A_{10}\omega = -[2(v^4-1)]^{-\frac{1}{2}}.$$

Since

$$(\lambda^2(\eta')^2+\eta'\eta''')' = -\Delta^2 A_{10}^2\omega^3(\lambda^2-\omega^2)\sin 2(\omega-\delta)\xi+O(\Delta^4)$$

the equation (5a) immediately gives

$$A_{20} = -\frac{(\lambda^2-\omega^2)\omega^3 A_{10}^2}{2[4\omega^2(\lambda^2-v^2)+\beta]} = -\frac{\omega A_{10}^2}{8}+O(\lambda^{-2})$$

$$B = \frac{\Delta^2 A_{10}}{(4\lambda^2\varkappa)}.$$

The constant B is selected to satisfy the discontinuity condition (3b) in the tangential force, which is

$$\begin{aligned}\frac{T}{EA} &= \gamma'+\tfrac{1}{2}(\eta')^2+O(\Delta^4)\\ &= \Delta^2\left[\frac{A_{10}^2\omega^2}{4}+\frac{A_{10}^2\omega^2}{4}\frac{4\omega^2(\omega^2-v^2)+\beta}{4\omega^2(\lambda^2-v^2)+\beta}\cos 2\omega\xi-(\operatorname{sgn}\xi)\frac{A_{10}}{4\lambda^2}e^{-\kappa|\xi|}\right]\\ &\quad+O(\Delta^4)\\ &= \frac{\Delta^2}{16(v^4-1)}[1+O(\lambda^{-2})]+O(\Delta^4)\end{aligned} \tag{13}$$

Hence the supercritical load speed produces an essentially constant axial force when λ is large.

The substitution of (12) into (5b), with the use of (13), gives for the coefficient of $\sin(\omega-\delta)\xi$

$$\Delta A_{10}[(\omega-\delta)^4-2v^2(\omega-\delta)^2+1]-\frac{\Delta^3 A_{10}^3\omega^3}{4}\left[3\mu\omega^5+\frac{3\alpha}{\omega^3}-2\lambda^2\omega+O(1)\right]+O(\Delta^5) = 0$$

Using the expansion for δ and the fact that $\omega^4-2v^2\omega^2+1 = 0$ gives

$$\delta_1 = \frac{2\lambda^2\omega^4-3\alpha-3\mu\omega^8+O(1)}{16\omega(\omega^2-v^2)}$$

where $O(1)$ denotes terms bounded by a constant independent of λ, α, or μ. The coefficient of $\sin 3(\omega-\delta)\xi$ from the substitution of (12) into (5b) gives

$$A_{30} = \frac{-9\mu\omega^8-\alpha+O(1)}{4(81\omega^4-18v^2\omega^2+1)}.$$

The exponential term in the expression for γ (12) induces a similar term in the expression for η. However, as in (13), such terms have a small effect.

The constant

$$A_{11} = \begin{cases} A_{11b} & \text{for} \quad \xi>0 \\ A_{11a} & \text{for} \quad \xi<0 \end{cases}$$

is determined from the condition that η' is continuous, i.e.,

$$(A_{11}\omega-\delta_1 A_{10}+3A_{30}\omega)_b = (A_{11}\omega-\delta_1 A_{10}+3A_{30}\omega)_a \tag{14a}$$

and from the condition (3*a*) which gives

$$(A_{11}\omega^3-3\delta_1 A_{10}\omega^2+27A_{30}\omega^3)_b = (A_{11}\omega^3-3\delta_1 A_{10}\omega^2+27A_{30}\omega^3)_a \tag{14b}$$

in which the terms of higher order in α, μ, and λ have been omitted. The solution for A_{11b} and A_{11a} is readily obtained from the pair of linear equations (14). For $v \gtrsim 2$, for which $\omega_b^2 = \omega_a^{-2} \approx 2v^2$, the result is especially simple

$$A_{11b} = \frac{A_{10}^3\omega^3}{(2v^2)^{\frac{1}{2}}}[-\tfrac{9}{8}\mu v^4+\tfrac{3}{4}\lambda^2+O(\alpha v^{-4})+O(1)]$$

$$A_{11a} = \frac{A_{10}^3\omega^3}{(2v^2)^{\frac{1}{2}}}[\tfrac{9}{8}\alpha v^4+\frac{\lambda^2}{4}+O(\mu v^{-4})+O(1)].$$

The maximum curvature, at $\xi = \pi/2(\omega_b-\delta)$, is for $v \gtrsim 2$

$$|\tau|_{\max} \approx \frac{1}{2^{\frac{1}{2}}\lambda}[\Delta A_{10}\omega^2+\Delta^3(-2A_{10}\delta_1\omega+A_{11}\omega^2-9A_{30}\omega^2)]_b$$

$$\approx \tau_0\left\{1+\tau_0^2[\tfrac{3}{2}\mu\lambda^2+\frac{\lambda^4}{8v^4}+O(\alpha v^{-6})]\right\} \tag{15a}$$

where τ_0 is the linear result (8b). The maximum deflection is at $\xi = -\pi/2(\omega_a-\delta)$ which is, for $v \gtrsim 2$,

$$\frac{\eta_{\max}}{2^{\frac{1}{2}}\lambda} = -\frac{\Delta A_{10}}{2^{\frac{1}{2}}\lambda}\left\{1+\left(\frac{\Delta A_{10}}{2^{\frac{1}{2}}\lambda}\right)^2\left[\frac{\alpha\lambda^2}{2}+\frac{\lambda^4}{8v^4}+O(\mu v^{-6})\right]\right\} \tag{15b}$$

Thus the nonlinearities due to large deformation, given by the parameter λ, and due to the softening beam material, given by μ, and the softening foundation, given by α, all tend to increase the curvature and deflection over that indicated by the linear result. Note that the foundation softening has little effect on the head wave-train (15a) while the material softening has little effect on the trailing long wavelength, large amplitude wave train (15b).

PERTURBATION SOLUTION FOR CRITICAL SPEED

A Lindstedt expansion is valid for the critical speed $v = 1$ when the foundation non-linearities dominate the geometric nonlinearity, i.e. when k is real, where k is the quantity

$$k = (3\alpha+3\mu-2\lambda^2)^{\frac{1}{2}}/4.$$

Instead of (12) the expansion is

$$\eta = \Delta^{\frac{1}{2}}A_1\sin(1-\delta)\xi+\Delta^{\frac{3}{2}}A_3\sin 3(\omega-\delta)\xi+\ldots$$

$$\gamma = \Delta A_2\sin 2(1-\delta)\xi+\ldots \tag{16}$$

where

$$A_i = A_{i0}+\Delta^{\frac{1}{2}}A_{i1}+\ldots$$

$$\delta = \Delta^{\frac{1}{2}}\delta_1+\Delta\delta_2+\ldots$$

The leading constant A_{10} is not known since no linear solution exists. However, the result for the axial force is similar to (13)

$$\frac{T}{EA} = \frac{\Delta A_{10}^2}{4}[1+O(\lambda^{-2})]+O(\Delta^{\frac{3}{2}})$$

where the discontinuity (3b) is in the term $O(\Delta^{\frac{3}{2}})$. The substitution of (16) into (5b) gives the coefficient of $\sin(1-\delta)\xi$

$$\Delta^{\frac{1}{2}}A_{10}[(1-\delta)^4-2(1-\delta)^2+1]-\Delta^{\frac{3}{2}}\frac{A_{10}^3}{4}[3\mu+3\alpha-2\lambda^2+O(1)]+O(\Delta^{\frac{5}{2}}) = 0.$$

The difference in the expansions (16) and (12) is due to the vanishing of the term linear in δ for $v = 1$. The result is

$$\delta_1 = \begin{cases} \delta_{1b} = kA_{10b} & \text{for} \quad \xi > 0 \\ \delta_{1a} = -kA_{10a} & \text{for} \quad \xi < 0. \end{cases}$$

The continuity of η' requires that

$$[\Delta^{\frac{1}{2}}A_{10}+\Delta(A_{11}-\delta_1 A_{10})]_b = [\Delta^{\frac{1}{2}}A_{10}+\Delta(A_{11}-\delta_1 A_{10})]_a$$

which is satisfied by

$$\begin{aligned} A_{10a} &= A_{10b} = A_{10} \\ A_{11b} &= \delta_{1b}A_{10} = kA_{10}^2 \\ A_{11a} &= \delta_{1a}A_{10} = -kA_{10}^2. \end{aligned} \tag{17}$$

The condition (3a) gives

$$-[\Delta^{\frac{1}{2}}A_{10}+\Delta(A_{11}-3\delta_1 A_{10})]_b+[\Delta^{\frac{1}{2}}A_{10}+\Delta(A_{11}-3\delta_1 A_{10})]_a = \Delta$$

which, with (17), gives

$$A_{10} = -(2k^{\frac{1}{2}})^{-1}.$$

The maximum curvature is at $\xi = \pi/2(1-\delta_b)$

$$|\tau|_{\max} = \left(\frac{\Delta}{8\lambda^2 k}\right)^{\frac{1}{2}}\left[1+\left(\frac{\Delta}{8\lambda^2 k}\right)^{\frac{1}{2}}2^{\frac{1}{2}}\lambda k+O(\Delta)\right].$$

An interesting situation develops when the geometric nonlinearity dominates, i.e. when

$$2\lambda^2 > 3\alpha+3\mu.$$

The Poincaré type of expansion (9) gives secular terms and the Lindstedt expansion (12) gives imaginary constants. Since the solution for $v < 1$ is exponential and for $v > 1$ is periodic, some type of transitional behavior might be expected at $v = 1$.

A suitable perturbation expansion which successfully avoids secular terms is

$$\eta = (f+A_3f^3+A_5f^5+\ldots)\sin\xi+(B_3f^3+B_5f^5+\ldots)\sin 3\xi+\ldots$$
$$+(A_2f^2+A_4f^4+\ldots)\cos\xi+(B_4f^4+\ldots)\cos 3\xi+\ldots+\varepsilon^4\eta_1+\ldots \quad (18a)$$
$$\gamma = C_3f^3+C_5f^5+\ldots+\varepsilon^3\gamma_1+\ldots$$
$$+(D_2f^2+D_4f^4+\ldots)\sin 2\xi+(E_4f^4+\ldots)\sin 4\xi+\ldots$$
$$+(D_3f^3+\ldots)\cos 2\xi+\ldots \quad (18b)$$

where A_i, B_i, C_i, and D_i are constants and where

$$f = \frac{\varepsilon}{1+\varepsilon k\xi} \quad (18c)$$

in which ε and k are real constants to be determined. The terms γ_i and η_i are of exponential behavior. For small ε, (18) is oscillatory but with a slowly changing monotonic envelope. The function f is convenient to use since its derivatives have a simple form

$$\frac{\mathrm{d}^nf}{\mathrm{d}\xi^n} = (-k)^n n!\, f^{n+1}.$$

The expressions (18) are substituted into (5a) and these results are obtained

$$C_3 = -\frac{k}{2\beta}(\lambda^2-1)$$
$$D_2 = -\frac{\lambda^2-1}{8(\lambda^2-1+\beta/4)}$$
$$D_3 = \frac{16kD_2(\lambda^2-1)-\lambda^2(3k+2A_2)+5k+2A_2}{8(\lambda^2-1+\beta/4)}.$$

Furthermore γ_1 is the complementary solution of (5a)

$$\gamma_1 = F_1\,\mathrm{e}^{-\kappa\xi}$$

where F_1 is a constant. The axial strain is

$$\frac{T}{EA} = \gamma'+\tfrac{1}{2}(\eta')^2+O(f^4)$$
$$= \mathrm{e}^3\gamma_1'+\frac{f^2}{4}+\frac{1+8D_2}{4}f^2\cos 2\xi-\left(2kD_2+2D_3+\frac{k+A_2}{2}\right)f^3\sin 2\xi+O(f^4)$$
$$= \mathrm{e}^3\gamma_1'+\frac{f^2}{4}+O(\lambda^{-2})f^2\cos 2\xi+k[1+O(\lambda^{-2})]f^2\sin 2\xi+O(f^4).$$

The constant A_2, which is unknown at this point, cancels from the leading term of the coefficient of $f^3 \sin 2\xi$. Next the expressions (18) are substituted into (5b) with the results

$$k = [(2\lambda^2 - 3\alpha - 3\mu)/32]^{\frac{1}{2}}$$

$$A_2 = -(2\lambda^2 - 9\alpha + 24\mu)/64k$$

$$A_3 = 3\lambda^4/40\beta$$

$$B_3 = -3\beta/2^{10} - (\alpha + 9\mu)/2^8$$

$$\eta_1 = -\frac{\lambda^2 F_1}{2\varkappa\varepsilon} f e^{-\kappa\xi}[\sin\xi + O(\lambda^{-1})].$$

Hence the first significant-terms in the expansion for η are

$$\eta \approx (f + A_3 f^3)\sin(\xi + \varphi) + A_2 f^2 \cos(\xi + \varphi) \tag{19}$$

where φ is a constant which provides an appropriate phase shift to satisfy the conditions at $\xi = 0$, while f is given by (18c). When ξ is replaced by $-\xi$ in (19), a solution of (5) which vanishes as $\xi \to -\infty$ is obtained. Thus (19) with ξ replaced by $|\xi|$ satisfies the conditions of boundedness and continuity of η and η'' (and γ'). The continuity of η' is satisfied if φ is chosen so that $\eta'(0) = 0$, i.e.

$$\varphi = \frac{\pi}{2} - \varepsilon(A_2 + k) + O(\varepsilon^3).$$

The continuity of γ is satisfied by choosing

$$F_1 = -C_3 - 2D_2(A_2 + k) + D_3 \approx k\lambda^2/2\beta.$$

The discontinuity condition (3a) gives

$$\eta'''(0) \approx 2k\varepsilon^2 \sin\varphi = \Delta/2.$$

Thus

$$\varepsilon = (\Delta/4k)^{\frac{1}{2}}$$

and the leading terms in the expressions for the maximum curvature and strain are

$$\tau_{\max} = \frac{\varepsilon}{2^{\frac{1}{2}}\lambda}[1 + O(\varepsilon^2)]$$

$$= (\Delta/8k\lambda^2)^{\frac{1}{2}}[1 + O(\Delta)]$$

$$(T/EA)_{\max} = \varepsilon^2/4 = \Delta/16k.$$

Thus the steady-state solution for the critical load speed indicates a response which varies with the square root of the load in either situation of the geometric or material nonlinearities dominant.

ELASTIC–PLASTIC SOLUTIONS

In this section the effects of plastic flow of the beam and foundation are considered. For simplicity the effects of the geometric nonlinearities are neglected, only supercritical load speeds $v > 1$ are considered, and the plastic flow of the beam and foundation are considered separately.

Plastic foundation

For the elastic beam on an elastic–perfectly plastic foundation the equation is

$$\eta'''' + 2v^2\eta'' + F(\eta) = \Delta\delta(\xi) \tag{20}$$

where the relation between the deflection and the foundation force is shown in Fig. 2 for a cycle of loading and unloading. A solution has the behavior indicated in Fig. 3. Ahead of the load is the short-wavelength elastic wave train. The plastic flow occurs a distance l_1 behind the load in a region of length $l_2 - l_1$. Behind the yielding region, the foundation has a permanent deformation η_1 to which is added the long-wavelength elastic wave train. Thus the solution is

$$\eta = \begin{cases} C_1 \sin \omega_b\xi & \text{for} \quad \xi > 0 \\ C_3 \sin \omega_b\xi + C_4 \cos \omega_b\xi + C_5 \sin \omega_a\xi + C_6 \cos \omega_a\xi & \text{for} \quad l_1 < \xi < 0 \\ -\dfrac{\eta_0}{4v^2}(\xi + l_2)^2 + \eta_0 + \eta_1 - C_8\{1 - \cos[\sqrt{2}v(\xi + l_2)]\} & \text{for} \quad -l_2 < \xi < -l_1 \\ \eta_1 + \eta_0 \cos[\omega_a(\xi + l_2)] & \text{for} \quad \xi < -l_2. \end{cases} \tag{21}$$

The constants must be chosen so that η, η', η'', η''' are continuous, except for the discontinuity at the load

$$\eta'''(0^+) - \eta'''(0^-) = \Delta$$

The above solution already satisfies the condition of zero velocity at the end of yielding $\eta'(-l_2) = 0$.

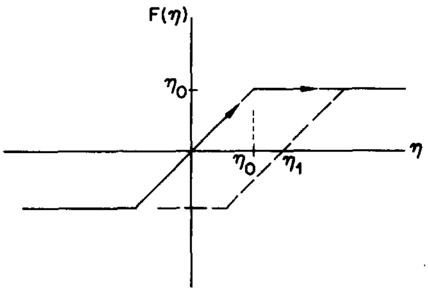

FIG. 2. Elastic–perfectly plastic foundation force.

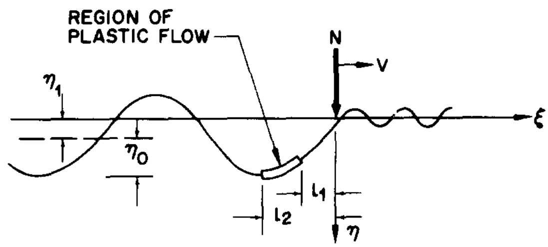

FIG. 3. Deformation of beam with supercritical load speed on an elastic–plastic foundation.

The conditions at the load $\xi = 0$ give the relations

$$C_6 = 0, \qquad C_2 = C_4$$

$$C_1 = C_3 + \frac{\omega_a}{\omega_b} C_5, \qquad C_5 = -\frac{\Delta}{\omega_a(\omega_b^2 - \omega_a^2)}.$$

Continuity at $\xi = -l_2$ is satisfied if

$$C_8 = \frac{\eta_0}{2v^2}\left(\omega_a^2 - \frac{1}{2v^2}\right).$$

This leaves five unknowns $C_3, C_4, \eta_1, l_1, l_2$ with which to satisfy the four continuity conditions and the condition that $\eta = \eta_0$ at the start of yielding $\xi = -l_1$. The conditions that $\eta = \eta_0$ and that η'' is continuous give

$$\begin{aligned} -C_3 \sin\omega_b l_1 + C_4 \cos\omega_b l_1 &= \eta_0 + C_5 \sin\omega_a l_1 \\ -C_3 \sin\omega_b l_1 + C_4 \cos\omega_b l_1 &= \frac{1}{\omega_b^2}\left[\frac{\eta_0}{2v^2} + C_8 2v^2 \cos\sqrt{2}v(l_2-l_1) - C_5\omega_a^2 \sin\omega_a l_1\right] \end{aligned} \tag{22}$$

The right-hand sides must be equal which gives

$$\begin{aligned} \nu \sin\omega_a l_1 &= 1 + \frac{\omega_a^2 - \dfrac{1}{2v^2}}{\omega_b^2 - \omega_a^2}[1 - \cos\sqrt{2}v(l_2 - l_1)] \\ &= 1 + O(v^{-8}) \end{aligned}$$

where

$$\nu = \frac{\Delta}{\omega_a(\omega_b^2 - \omega_a^2)\eta_0}$$

which is the ratio of the maximum deflection of the elastic solution (8) to the yield deformation of the foundation η_0. Thus for $v \gtrsim 2$

$$\begin{aligned} \sin\omega_a l_1 &= \nu^{-1} \\ l_1 &= \sqrt{2}v \sin^{-1}\nu^{-1}. \end{aligned}$$

Similarly from the continuity of η' and η''' comes the relations

$$\begin{aligned} C_3 \cos\omega_b l_1 + C_4 \sin\omega_b l_1 &= -\frac{1}{\omega_b}\Big\{C_5\omega_a \cos\omega_a l_1 \\ &+ \frac{\eta_0}{2v^2}(l_2 - l_1) + C_8\sqrt{2}v \sin[\sqrt{2}v(l_2 - l_1)]\Big\} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \nu\cos\omega_a l_1 &= \omega_a(l_2 - l_1)\frac{\omega_b^4}{2v^2(\omega_b^2 - \omega_a^2)} \\ &\times\left[1 + \frac{2v^2\left(\omega_a^2 - \dfrac{1}{2v^2}\right)(\omega_b^2 - 2v^2)}{\omega_b^2}\frac{\sin[\sqrt{2}v(l_2 - l_1)]}{\sqrt{2}v(l_2 - l_1)}\right] \\ &= \omega_a(l_2 - l_1)[1 + O(v^{-4})]. \end{aligned}$$

Thus for $v \gtrsim 2$

$$\omega_a(l_2-l_1) = v\cos\omega_a l_1$$
$$= \cot\omega_a l_1 = (v^2-1)^{\frac{1}{2}}.$$

The right-hand side of the above equation (22) and (23) may now be evaluated

$$C_3\cos\omega_b l_1 + C_4\sin\omega_b l_1 = \eta_0\frac{\omega_a(l_2-l_1)}{2v^2\omega_b^2(\omega_b^2-\omega_a^2)} \times\left\{1-\frac{2v^2\left(\omega_a^2-\dfrac{1}{2v^2}\right)(2v^2-\omega_a^2)\sin[\sqrt{2}v(l_2-l_1)]}{\omega_a^2\sqrt{2}v(l_2-l_1)}\right\}$$
$$= \eta_0(v^2-1)^{\frac{1}{2}}O(v^{-5})$$

$$-C_3\sin\omega_b l_1 + C_4\cos\omega_b l_1 = -\eta_0\frac{\omega_a^2-\dfrac{1}{2v^2}}{\omega_b^2-\omega_a^2}\{1-\cos[\sqrt{2}v(l_2-l_1)]\}$$
$$= \eta_0 O(v^{-8}).$$

Therefore, for $v \gtrsim 2$, C_3 and C_4 are negligible

$$C_3, C_4 = \eta_0(v^2-1)^{\frac{1}{2}}O(v^{-5}).$$

The constants C_1 and C_2 are

$$C_1 = C_3+\frac{\omega_a}{\omega_b}C_5 = -\frac{\Delta}{\omega_b(\omega_b^2-\omega_a^2)}+\eta_0(v^2-1)^{\frac{1}{2}}O(v^{-5})$$
$$= \frac{\omega^a}{\omega_b}C_5[1+O(v^{-3})]$$
$$C_2 = C_4 = \eta_0(v^2-1)^{\frac{1}{2}}O(v^{-5}).$$

The only remaining constant to be evaluated is η_1. From the condition that $\eta(-l_1) = \eta_0$,

$$\eta_1 = \frac{\eta_0}{4v^2}(l_2-l_1)^2 + C_8\{1-\cos[\sqrt{2}v(l_2-l_1)]\}$$
$$= \eta_0\frac{v^2-1}{2}[1+O(v^{-4})].$$

Thus the solution (21) for $v \gtrsim 2$ is essentially

$$\eta = \begin{cases} -\dfrac{\Delta}{(\sqrt{2}v)^3}\sin(\sqrt{2}v\xi) & \text{for} \quad \xi > 0 \\ -\dfrac{\Delta}{\sqrt{2}v}\sin\dfrac{\xi}{\sqrt{2}v} & \text{for} \quad -l_1 < \xi < 0 \\ \eta_0+\eta_1-\dfrac{\eta_0}{4v^2}(\xi+l_2)^2 & \text{for} \quad -l_2 < \xi < -l_1 \\ \eta_1+\eta_0\cos\left[\dfrac{1}{\sqrt{2}v}(\xi+l_2)\right] & \text{for} \quad \xi < -l_2 \end{cases} \tag{24}$$

where

$$l_1 = \sqrt{2v}\sin^{-1}\nu^{-1} \tag{25a}$$

$$l_2 - l_1 = \sqrt{2v}(\nu^2-1)^{\frac{1}{2}} \tag{25b}$$

$$\nu = \frac{\Delta}{\sqrt{2v}\eta_0} \tag{25c}$$

$$\eta_1 = \eta_0\frac{\nu^2-1}{2}. \tag{25d}$$

Thus the bending wave ahead of the load is unaffected by the yielding of the foundation behind the load.

For a comparison, if the impulse due to the moving load acting on each beam element would act on all the beam elements simultaneously, then the beam will behave as a simple mass on an elastic–plastic spring. The permanent deformation thus obtained is exactly the same as in (24).

Plastic beam

The case of an elastic foundation with a beam, whose moment–curvature relation is the elastic–perfectly plastic relation shown in Fig. 4, is now considered. The equation is

$$m'' + 2v^2\eta'' + \eta = \Delta\delta(\xi). \tag{26}$$

Since a permanent curvature left in the beam by the load would cause unbounded displacement at $\xi = -\infty$, the path in Fig. 4 for a given beam element must be a closed cycle. Thus one is led to the type of solution indicated in Fig. 5 in which the yielding occurs at the two points $\xi = l_1$ and l_2. Plastic flow in an interval is not possible, since in the interval m would be constant, leaving the solution of (26) for the interval with only two arbitrary constants. Four arbitrary constants are required to satisfy continuity conditions. Since rotary inertia is neglected, a discontinuity in curvature is permissible. For continuity of the transverse shear, η''' must be continuous. Thus at $\xi = l_1$ and l_2 the quantities η, η', η''' are continuous while

$$\eta''(l_2^+) = -m_0$$
$$\eta''(l_2^-) = -m_0 - m_1$$
$$\eta''(l_1^+) = m_0 - m_1$$
$$\eta''(l_1^-) = m_0$$
$$\eta'''(l_2) = 0.$$

The solution is

$$\eta = \begin{cases} \dfrac{m_0}{\omega_b^2}\cos\omega_b(\xi - l_2) & \text{for} \quad l_2 < \xi \\ C_1\cos\omega_a(\xi - l_2) + C_2\cos\omega_b(\xi - l_2) & \text{for} \quad l_1 < \xi < l_2 \\ C_3\cos\omega_a\xi + C_4\sin\omega_a\xi + C_5\sin\omega_b\xi & \text{for} \quad 0 < \xi < l_1 \\ C_3\cos\omega_a\xi + C_6\sin\omega_a\xi & \text{for} \quad \xi < 0 \end{cases} \tag{27}$$

where the zero constants have been omitted. The conditions at $\xi = 0$ give

$$C_5 = -\frac{\Delta}{\omega_b(\omega_b^2-\omega_a^2)}$$

$$C_6 = C_4+\frac{\omega_b}{\omega_a}C_5.$$

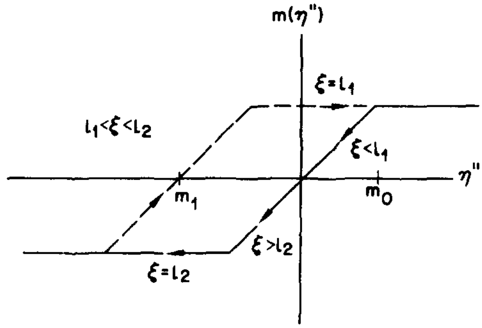

FIG. 4. Elastic–perfectly plastic beam moment–curvature relation.

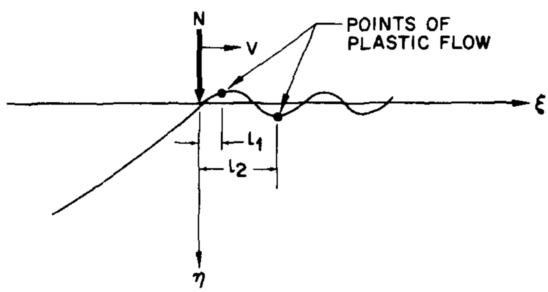

FIG. 5. Deformation of elastic–perfectly plastic beam on an elastic foundation with supercritical load speed.

The conditions at $\xi = l_2$ give

$$C_1 = -\frac{m_1}{\omega_b^2-\omega_a^2}$$

$$C_2 = \frac{m_1}{\omega_b^2-\omega_a^2}+\frac{m_0}{\omega_b^2}.$$

The conditions at $\xi = l_1$ give the relations

$$\varphi \sin \omega_b l_1 = 1+\frac{\omega_a^2}{\omega_b^2-\omega_a^2}[1+(1+\theta)\cos \omega_b(l_2-l_1)-\theta \cos \omega_a(l_2-l_1)] \tag{28a}$$

$$= \theta-(1+\theta)\cos(\omega_b(l_2-l_1)) \tag{28b}$$

$$\varphi \cos \omega_b l_1 = -(1+\theta)\sin(\omega_b(l_2-l_1)) \tag{28c}$$

$$C_3 = \frac{m_0}{\omega_b^2}\theta \sin(\omega_a(l_2 - l_1)) \sin \omega_a l_1 + \frac{m_0}{\omega_a^2}(\varphi \sin \omega_b l_1 - 1) \cos \omega_a l_1 \tag{28d}$$

$$C_4 = \frac{m_0}{\omega_a^2}(\varphi \sin \omega_b l_1 - 1) \sin \omega_a l_1 - \frac{m_0}{\omega_b^2}\theta \sin \omega_a(l_2 - l_1) \cos \omega_a l_1 \tag{28e}$$

where φ is the ratio of the maximum bending moment of the elastic solution (8) to the yield moment

$$\varphi = \frac{\omega_b \Delta}{(\omega_b^2 - \omega_a^2)m_0}$$

and θ is

$$\theta = \frac{m_1 \omega_b^2}{m_0(\omega_b^2 - \omega_a^2)}$$

which involves the ratio m_1/m_0 of the jump in curvature due to plastic flow to the curvature at first yielding.

The three relations (28a, b, c) determine the three unknowns θ, l_1, and l_2. Note that when $\varphi = 1$ exactly, the solution (8) is obtained. Generally, for $\theta > 1$, the evaluation is difficult; however for $v \gtrsim 2$, (28a) is

$$\varphi \sin \omega_b l_1 = 1 + 0((1+\theta)/v^4).$$

Hence, if $\theta \lesssim 1$,

$$\omega_b l_1 = \sin^{-1} \varphi^{-1} \qquad (0 < \omega_b l_1 \leq \pi/2). \tag{29a}$$

Then (27c) gives

$$\sin \omega_b(l_2 - l_1) = -\frac{(\varphi^2 - 1)^{\frac{1}{2}}}{1+\theta} \qquad \left(\pi \leq \omega_b(l_2 - l_1) < \frac{3\pi}{2}\right) \tag{29b}$$

with which θ can be obtained from (28b)

$$\theta = (\varphi^2 - 1)/4 \tag{29c}$$

Since for $v \gtrsim 2$

$$\varphi = \frac{\Delta}{2^{\frac{1}{2}} m_0 v}$$

$$\theta = m_1/m_0$$

(29c) provides a simple relation between the load magnitude, the yield curvature and the plastic flow. The remaining constants C_3 and C_4 are $O(m_0/v^2)$. Thus the solution (26) is virtually the same as (8) for $\xi < 0$.

Plastic beam and plastic foundation

The load moving at a supercritical speed generates the long-wavelength train behind and the short-wavelength train ahead of the load, according to the elastic solution (8). The largest deflection occurs behind the load while the largest bending occurs ahead of the load. Thus the plastic flow of the foundation, considered separately, occurs behind the load and does not affect the bending wave ahead of the load (24), while the plastic flow of the beam is bending, considered separately, occurs ahead of the load but does not affect the wave behind the load (26). Therefore if plastic flow of both beam and foundation are considered simultaneously, the results (25) and (29) should not be significantly changed.

CLOSURE

From the preceding results the following conclusions are made.

The steady-state response varies with the square root of the load magnitude for the critical load speed and so, for sufficiently small load magnitudes, is large in comparison with the response for noncritical load speeds. Thus for the usual situation, the (undamped) linear result for the transient behavior, as in [2], should be valid until the magnitude approaches the smaller of the linear steady-state solution with (small) damping, as in [1], or the present nonlinear steady-state solution.

For large load magnitudes, the first nonlinear correction terms indicate a maximum strain larger than the linear result for all (noncritical) load speed, at least for the usual situation of positive values of α and μ, corresponding to softening elastic beam and foundation materials. Stiffening materials, with α and μ negative, tend to decrease the maximum curvature.

For plasticity effects for supercritical load speeds, the permanent deformation of the foundation is essentially the same as when the entire beam is instantaneously given an impulse equal to N/V. No permanent bending deformation occurs. A closed cycle on the moment-curvature diagram is transversed as the load approaches and then passes each element of the beam.

Acknowledgements—A solution for a rigid–perfectly plastic beam, which led to the present elastic–plastic solution, was suggested by G. B. Cline in informal discussion. Several errors in the basic equations were corrected by the referees, one of whom brought to my attention Reference [5].

REFERENCES

[1] J. T. KENNEY, Jr., Steady-state vibrations of beam on elastic foundation for moving loads. *J. appl. Mech.* **21**, 359–364 (1954).
[2] C. R. STEELE, The finite beam with a moving load. *J. appl. Mech.* **34**, 111–118 (1967).
[3] J. J. STOKER, *Water Waves*. Interscience (1957).
[4] L. CESARI, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*. Springer (1963).
[5] I. TADJBAKHSH, The variational theory of the plane notion of the extensible elastica, *Int. J. engng. Sci.* **4**, 433–450 (1966).

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Résumé—La solution "régime permanent" des équations rendues linéaires pour une poutre sur une fondation élastique (sans amortissment) avec une charge se déplaçant à une certaine vitesse, appelée vitesse "critique", n'existe pas. Dans cet exposé, des solutions de perturbation qui conviennent sont obtenus pour des équations pour le mouvement à régime permanent qui comprennent des expressions non linéaires géométriques et relatives au matériau. Des termes séculaires sont évités en utilisant l'expansion habituelle de Poincarré pour les vitesses de charge inférieures aux vitesses critiques et une expansion de Lindstedt pour des vitesses supérieures aux vitesses critiques qui peuvent être étendues à la vitesse critique quand la condition non linéaire du matériau domine. Toutefois aucune des deux expansions n'est valide pour la vitesse critique quand la condition non linéaire géométrique domine, comme dans le cas d'une poutre très frêle. Pour ce cas, une expansion satisfaisante se trouve qui donne une solution qui est principalement périodique (par rapport à la distance de la charge) mais avec un lent décroissement monotonique dans l'enveloppe. Les résultats sont les premières corrections non linéaires des solutions linéaires pour des vitesses de charge non critiques et la solution pour la vitesse critique qui donne un effort qui varie avec la racine carrée de la charge. Pour indiquer l'effet de plasticité, une solution simple pour des vitesses de charge supérieures aux vitesses critiques pour des matériaux pour poutres plastiques élastiques et pour foundations est aussi obtenu. La solution consiste principalement dans les ondes élastiques mais avec une région d'écoulement plastique de la fondation derrière la charge et deux points de limite élastique de la poutre en pliant en avant de la charge.

Zusammenfassung—Die "Beharrungszustands"-Lösung der linearisierten Gleichungen eines Balkens auf elastischer Grundlage (ohne Dämpfung) mit einer Belastung die sich mit einer Geschwindigkeit bewegt die als "kritisch" bezeichnet wird, gibt es nicht. In dieser Arbeit werden entsprechende Störungslösungen erhalten, die für Gleichungen des Beharrungszustandes gelten, die geometrische und materielle Unregelmässigkeiten enthalten. Säkuläre Ausdrücke werden vermieden, durch Verwendung der üblichen Poincare Expansionen für sub-kritische Lastgeschwindigkeiten, sowie Lindstedt Expansionen für überkritische Geschwindigkeiten die in die kritische Geschwindigkeit ausegedehnt werden können, wenn di Material-Nichtlinearität überwiegt. Keine der Expansionen gilt aber, für die kritische Geschwindigkeit wenn die geometrische Nichtlinearität überwiegt, wie bei enem schlanken Balken. Für diesen Fall wird eine Expansion gefunden, mit einer Lösung die hauptsächlich periodisch ist (abhängig von der Entfernung der Last) aber mit einer langsamen monotonen Abnahme der Hüllkurve. Die Resultate sind die ersten nichtlinearen Korrekturne der Linearlösungen für nichtkritische Lastgeschwindigkeiten und Lösung der kritischen Geschwindigkeit, eine Spannung mit der Quadratwurzel der Last variert. Um den Effekt der Plastizität anzuzeigen wird auch eine einfache Lösung der überkritischen Lastgeschwindigkeiten elastoplastischer Balken und Grundlagen-Materialien erhalten. Diese Lösung besteht hauptsächlich aus den elastischen Wellen, aber der plastische Bereich der Grundlage ist hinter der Belastung und die zwei Fliesspunkte des Balkens in Biegung sind vor der Belastung.

Абстракт—"Устойчивое" решение линеаризованных уравнений для балки на упругомо сновании (без демпфирования) с грузом, движущимся с некоторой скоростью, отнесенное как "критическая" скорость, совершенно не возможно. В этой работе получено удовлетворительное решение возмущений для уравнений устойчивости движения, которые заключают геометрические и материальные нелинейности. Удалось обойти вековые члены используя обычные разложения Пуанкаре для скоростей ниже критической, а разложеняе Линдстедта для скоростей сверхкритических, которые могут быть расширены на критическую скорость, в случае когда преобладает материальная нелинейность. Тем не менее никакие разложения неважные для критических скоростей, в случае когда преобладает геометрическая нелинейность, а также для очень гибкой балки. В этом случае найдены удовлетворительные разложения, которые дают, в основном, периодические решения (лрц расстояни от груза), но с медленным уменьшением на огибающей. Результаты являются первой нелинейной поправкой лииейных решений для скоростей с некритической нагрузкой, а также для решения критической скорости, которая вызывает перемещение, изменяющееся с квадратным корнем нагрузки. Для указания пластического эффекта, получено также простое решение, касающееся скоростей сверхкритической нагрузки для упругопластической балки и материалов основания. Решение состоит, главным образом, из упругих волн, но с районом пластического течения основания за нагрузкой и двух пречелоь текучести балки, при изгибе, до нагрузки.